

A Note on the Critical Problem for Matroids

TAKAO ASANO, TAKAO NISHIZEKI, JAMES OXLEY AND NOBUJI SAITO

Let M be a matroid representable over $\text{GF}(q)$ and S be a subset of its ground set. In this note we prove that S is maximal with the property that the critical exponent $c(M|S; q)$ does not exceed k if and only if S is maximal with the property that $c(M \cdot S) \leq k$. In addition, we show that, for regular matroids, the corresponding result holds for the chromatic number.

The critical problem for matroids has been discussed in detail by several authors including Crapo and Rota [4, Chapter 16] and Welsh [10, Section 15.5]. In this note we shall in general follow the latter's terminology. In particular, if M is a matroid having rank function ρ and ground set $E(M)$, then its *chromatic polynomial* $P(M; \lambda)$ is defined by

$$P(M; \lambda) = \sum_{A \subseteq E(M)} (-1)^{|A|} \lambda^{\rho M - \rho A}.$$

Now suppose that M is representable over $\text{GF}(q)$ and that $\rho M = r$. Let ψ be a representation of M in $V(r, q)$, the r -dimensional vector space over $\text{GF}(q)$. Then for j in \mathbb{Z}^+ , we have, by [4, p. 16.4], that $P(M; q^j)$ equals the number of j -tuples (f_1, f_2, \dots, f_j) of linear functionals on $V(r, q)$ such that, for all e in $\psi(E(M))$, $f_i(e)$ is non-zero for some i . Thus, for all j in \mathbb{Z}^+ , $P(M; q^j)$ is non-negative. Furthermore, if $P(M; q^j)$ is positive, so is $P(M; q^{j+1})$. The *critical exponent* $c(M; q)$ of M is defined by

$$c(M; q) = \begin{cases} \infty, & \text{if } M \text{ has a loop;} \\ \min\{j \in \mathbb{Z}^+ : P(M; q^j) > 0\}, & \text{otherwise.} \end{cases}$$

The main result of this note is the following.

THEOREM 1. *Let M be a matroid representable over $\text{GF}(q)$ and S be a subset of $E(M)$. Then the following statements are equivalent.*

- (a) S is maximal with the property that $c(M|S; q) \leq k$.
- (b) S is maximal with the property that $c(M \cdot S; q) \leq k$.
- (c) S is maximal with the property that M has a minor having ground set S and critical exponent not exceeding k .

This result may be proved by using deletion–contraction arguments for the chromatic polynomial. However, the following approach, suggested by the referee, seems more enlightening. We shall use Tutte's theory of chain-groups (see [7] or [8]) recalling first the basic definitions. Let R be an integral domain and E be a finite set. A *chain on E over R* is a mapping of E into R . A *chain-group on E over R* is a set of chains on E over R that is closed under the operations of addition and multiplication by an element of R . The *support* $\sigma(f)$ of a chain f is $\{e \in E : f(e) \neq 0\}$. It can be shown (see, for example, [8, 1.22]) that if N is a chain-group, then the set of minimal non-empty supports of members of N is the set of circuits of a matroid $M(N)$ on E .

Now, as before, let M be a rank- r matroid representable over $\text{GF}(q)$ and let ψ be a representation of M in $V(r, q)$. Suppose that $|E(M)| = n$ and let A be the $r \times n$ matrix over $\text{GF}(q)$ whose columns are the vectors in $\{\psi(e) : e \in E(M)\}$. We now identify the elements of M with the corresponding columns of A . Then each row of A may be viewed

as a chain on $E(M)$ over $\text{GF}(q)$. The *chain-group* N generated by A is the set of chains which correspond to linear combinations of the rows of A . The matroid $M(N)$ of this chain-group is dual to M (see, for example, [11, Section 9.4]). Moreover, it is a routine exercise in linear algebra to verify that the restriction to $E(M)$ of any linear functional on $V(r, q)$ is a member of N and that every member of N can be obtained in this way. Using this, it follows directly from the definition of the critical exponent that

$$\begin{aligned} c(M; q) \leq k \text{ if and only if } N \text{ contains } k \text{ chains } f_1, f_2, \dots, f_k \\ \text{such that } E(M) = \bigcup_{i=1}^k \sigma(f_i). \end{aligned} \quad (1)$$

The next lemma is the key to the proof of Theorem 1. It relies on the following standard results for chain-groups. For a subset S of $E(M)$, let $N|S$ denote the set of restrictions to S of chains in N and let $N \cdot S$ denote the set of restrictions to S of chains f in N for which $\sigma(f) \subseteq S$. Then the matroids $M(N|S)$ and $M(N \cdot S)$ are dual to $M|S$ and $M \cdot S$, respectively [8, 2.31]. Now, with M and N as specified above, we have the following

LEMMA 2. *Suppose that $S \subseteq T \subseteq E(M)$. Then*

- (a) $c(M|S; q) \leq k$ if and only if N contains k chains f_1, f_2, \dots, f_k such that $S \subseteq \bigcup_{i=1}^k \sigma(f_i)$;
- (b) $c(M \cdot S; q) \leq k$ if and only if N contains k chains f_1, f_2, \dots, f_k such that $S = \bigcup_{i=1}^k \sigma(f_i)$;
and
- (c) $c((M|T) \cdot S; q) \leq k$ if and only if $N|T$ contains k chains f'_1, f'_2, \dots, f'_k such that $S = \bigcup_{i=1}^k \sigma(f'_i)$.

PROOF. Parts (a) and (b) are obtained by combining observation (1) with the above facts about chain-groups and their minors. Part (c) is obtained by applying (b) to $M|T$.

The next five observations are immediate consequences of this lemma. In each, M denotes an arbitrary matroid representable over $\text{GF}(q)$. If such a matroid has critical exponent one it is called *affine* since its underlying simple matroid is a submatroid of an affine geometry.

[6, Proposition 3.7]. If T is a non-empty subset of $E(M)$, then $c(M|T; q) \leq c(M \cdot T; q)$. (2)

If $c(M|S; q) \leq k$, then there is a set T containing S such that $c(M \cdot T; q) \leq k$. (3)

$c(M|S; q) \leq k$ if and only if there is a set T containing S such that $c((M|T) \cdot S; q) \leq k$. (4)

$$c(M; q) = \min\{n \in \mathbb{Z}^+ : E(M) = \bigcup_{i=1}^n S_i \text{ and } M|S_i \text{ is affine for all } i\}. \quad (5)$$

$$c(M; q) = \min\{n \in \mathbb{Z}^+ : E(M) = \bigcup_{i=1}^n S_i \text{ and } M \cdot S_i \text{ is affine for all } i\}. \quad (6)$$

The last two of these generalize results of Lindström [5, Theorem 14] and the author [6, Proposition 3.9] for binary matroids.

PROOF OF THEOREM 1. The equivalence of (a) and (c) is an immediate consequence of observation (4). To show that (a) implies (b), suppose that S is maximal with the property that $c(M|S; q) \leq k$. Then, by observation (3), $c(M \cdot T; q) \leq k$ for some set T containing S . By observation (2) and the choice of S , $T = S$ and (b) now follows easily.

We now show that (b) implies (a). Let S be a maximal set for which $c(M \cdot S; q) \leq k$. Then by observation (2), $c(M|S; q) \leq k$ and, by observation (3), S is maximal with this property. This completes the proof of the theorem.

A matroid with no odd circuits is called *bipartite*. It is well-known that a binary matroid M is bipartite if and only if it is affine (see, for example, [2, Theorem 10.3]). The next result follows immediately on combining this fact with Theorem 1 and (5).

COROLLARY 3. *Let M be a binary matroid and S be a subset of $E(M)$. Then the following statements are equivalent.*

- (a) S is maximal with the property that $M|S$ can be covered by k bipartite restrictions.
- (b) S is maximal with the property that $M \cdot S$ can be covered by k bipartite restrictions.
- (c) S is maximal with the property that M has a minor M' with ground set S such that M' can be covered by k bipartite restrictions.

By observation (6) each of the statements in Corollary 3 is equivalent to the three statements which are obtained by replacing 'bipartite restrictions' by 'bipartite contractions' in (a), (b) and (c).

A matroid is *Eulerian* if its ground set can be partitioned into circuits. Welsh [10] gave a matroid extension of a well-known graph-theoretic result by showing that a binary matroid is bipartite if and only if its dual is Eulerian. Using this, together with the above observations concerning bipartite contractions, it follows that the preceding corollary remains true if one inserts 'Eulerian' in place of 'bipartite' throughout the statement of the result.

By analogy with graphs, Welsh [11, p. 262] has defined the *chromatic number* $\chi(M)$ of a loopless matroid M to be

$$\min\{j \in \mathbb{Z}^+ : P(M; j) > 0\}.$$

He notes, however, that many fundamental properties of the chromatic number for graphs do not hold for this quantity. In particular, $P(M; j)$ may vanish or take negative values at integers j exceeding $\chi(M)$. This prompted consideration in [6] and [9] of the quantity $\pi(M)$ where

$$\pi(M) = \min\{j \in \mathbb{Z}^+ : P(M; j+k) > 0 \text{ for } k = 0, 1, 2, \dots\}.$$

Evidently, for graphic matroids M , $\chi(M) = \pi(M)$ and, in fact, it is an easy consequence of a result of Tutte [7, 5.44] that this remains true for arbitrary regular matroids. For such matroids, we have the following analogue of Theorem 1.

THEOREM 4. *Let M be a regular matroid and S be a subset of $E(M)$. Then the following statements are equivalent.*

- (a) S is maximal with the property that $\chi(M|S) \leq k$.
- (b) S is maximal with the property that $\chi(M \cdot S) \leq k$.
- (c) S is maximal with the property that M has a minor having ground set S and chromatic number not exceeding k .

The proof of Theorem 4 is similar to the proof of Theorem 1 but requires more of the theory of chain-groups. Let N be a chain-group on E over \mathbb{Z} . A chain f is *elementary* if $\sigma(f)$ is a circuit in $M(N)$. Such a chain is *primitive* if $f(e) \in \{-1, 0, 1\}$ for all e in E . A *regular* chain-group is a chain-group over \mathbb{Z} in which every elementary chain is a multiple of a primitive chain. Every regular matroid is dual to the matroid $M(N)$ of a regular chain-group [8, p. 4].

Let k be an integer exceeding one and N be a regular chain-group on a set E . If $f \in N$ and $\alpha \in \mathbb{Z}_k$, $\alpha \cdot f$ will denote the mapping from E into \mathbb{Z}_k defined by

$$(\alpha \cdot f)(e) = \alpha \cdot f(e)$$

where \mathbb{Z}_k is viewed here as a right \mathbb{Z} -module.

A mapping ξ from E into \mathbb{Z}_k is a \mathbb{Z}_k -chain of N if

$$\xi = \sum_{i=1}^m \alpha_i \cdot f_i$$

for some subset $\{f_1, f_2, \dots, f_m\}$ of N and some collection $\alpha_1, \alpha_2, \dots, \alpha_m$ of elements of \mathbb{Z}_k . Such a mapping is a *proper* \mathbb{Z}_k -chain of N if its support, $\{e \in E: \xi(e) \neq 0\}$, is E . Now let M be the dual of $M(N)$. Crapo [3, Theorem III] has shown that the number of proper \mathbb{Z}_k -chains of N equals $P(M; k)$ (see also [1]). Using this, the following analogue of Lemma 2 is straightforward.

LEMMA 5. *Let N be a regular chain-group on E and M be the dual of $M(N)$. Suppose that $S \subseteq T \subseteq E$. Then, for all integers k exceeding one,*

- (a) $\chi(M|S) \leq k$ if and only if there is a \mathbb{Z}_k -chain of N whose support contains S ;
- (b) $\chi(M \cdot S) \leq k$ if and only if there is a \mathbb{Z}_k -chain of N whose support equals S ;
- (c) $\chi((M|T) \cdot S) \leq k$ if and only if there is a \mathbb{Z}_k -chain of $N|T$ whose support equals S .

Using this lemma, one can easily derive the following analogues of observations (2)–(4). Theorem 4 can then be proved by mimicking the proof of Theorem 1. In each of the next three statements, M will denote an arbitrary regular matroid. If such a matroid has a loop, then we define its chromatic number to be ∞ .

If T is a non-empty subset of $E(M)$, then $\chi(M|T) \leq \chi(M \cdot T)$. (7)

If $\chi(M|S) \leq k$, then there is a set T containing S such that $\chi(M \cdot T) \leq k$.

$\chi(M|S) \leq k$ if and only if there is a set T containing S such that $\chi((M|T) \cdot S) \leq k$.

It is natural to try to extend Theorem 4 to larger classes of matroids. However, for binary matroids, (a), (b) and (c) need not be equivalent. To see this, let M be the Fano matroid and S be any 6-element subset of $E(M)$. Then $M \cdot S$ can be obtained from a 3-circuit by replacing each element by a pair of elements in parallel. Thus S is a maximal set for which $\chi(M \cdot S) \leq 3$. However, $M|S \cong M(K_4)$ so $\chi(M|S) = 4$. We note that, since $\chi(M') = \pi(M')$ for all minors M' of the Fano matroid, the same example shows that even if one replaces χ by π in (a)–(c), these statements still need not be equivalent for binary matroids.

A regular matroid is representable over all fields and hence over all finite fields $\text{GF}(q)$. It was noted above that for such a matroid M , $\chi(M) = \pi(M)$. Using this and the definitions of $\chi(M)$, $\pi(M)$ and $c(M; q)$, it is easy to show that

$$q^{c(M; q)-1} < \chi(M) = \pi(M) \leq q^{c(M; q)}.$$

For an arbitrary matroid M representable over $\text{GF}(q)$, considerably less is true. One still has that

$$q^{c(M; q)-1} < \pi(M) \quad \text{and} \quad \chi(M) \leq q^{c(M; q)}.$$

However, it may no longer be true that $\chi(M) > q^{c(M; q)-1}$, that $\chi(M) = \pi(M)$, or that $\pi(M) \leq q^{c(M; q)}$. To see the first two of these, let $M = \text{PG}(2, q) \oplus \text{PG}(2, q)$. Then $P(M; \lambda) = (\lambda - 1)^2(\lambda - q)^2(\lambda - q^2)^2$ and so

$$\chi(M) = \begin{cases} 2, & \text{if } q > 2, \\ 3, & \text{if } q = 2. \end{cases}$$

However, $\pi(M) = q^2 + 1$ and $c(M; q) = 3$.

As an example of a matroid M for which $\pi(M) > q^{c(M;q)}$, take $AG(4, q)$. This has critical exponent one and chromatic polynomial

$$P(AG(4, q); \lambda) = (\lambda - 1)(\lambda^4 - (q^4 - 1)\lambda^3 + (q^4 - 1)(q^3 - 1)\lambda^2 - (q^4 - 1)(q^3 - 1)(q^2 - 1)\lambda + (q^4 - 1)(q^3 - 1)(q^2 - 1)(q - 1)),$$

where the latter can be calculated using Rota's expansion for the chromatic polynomial in terms of the Möbius function (see, for example, [11, Theorem 15.3.1]). It is not difficult to check that, for all q , $P(AG(4; q); q + 1) < 0$, hence

$$\pi(AG(4, q)) \geq q + 2 > q^{c(AG(4,q);q)}.$$

REFERENCES

1. D. K. Arrowsmith and F. Jaeger, On the enumeration of chains in regular chain-groups, *J. Combin. Theory Ser. B* **32** (1982), 75–89.
2. T. H. Brylawski, A decomposition for combinatorial geometries, *Trans. Amer. Math. Soc.* **171** (1972), 235–282.
3. H. H. Crapo, The Tutte polynomial, *Aequationes Math.* **3** (1969), 211–229.
4. H. H. Crapo and G.-C. Rota, *On the Foundations of Combinatorial Theory: Combinatorial Geometries*, Preliminary ed., M.I.T. Press, Cambridge, Massachusetts, 1970.
5. B. Lindström, *Lectures on Matroids*, Lecture notes, University of Stockholm, 1974.
6. J. G. Oxley, Colouring, packing and the critical problem, *Quart. J. Math. Oxford* **29**(2) (1978), 11–22.
7. W. T. Tutte, Lectures on matroids, *J. Res. Nat. Bur. Standards Sect. B* **69B** (1965), 1–47.
8. W. T. Tutte, *Introduction to the Theory of Matroids*, American Elsevier, New York, 1971.
9. P. N. Walton and D. J. A. Welsh, On the chromatic number of binary matroids, *Mathematika* **27** (1980), 1–9.
10. D. J. A. Welsh, Euler and bipartite matroids, *J. Combin. Theory* **6** (1969), 375–377.
11. D. J. A. Welsh, *Matroid Theory*, *London Math. Soc. Monographs No. 8*, Academic Press, London, 1976.

Received 30 March 1981 and in revised form 15 November 1982

TAKAO ASANO

Department of Mathematical Engineering and Instrumentation Physics, Faculty of Engineering,
University of Tokyo, Bunkyo-ku, Tokyo, Japan 113,

TAKAO NISHIZEKI AND NOBUJI SAITO

Department of Electrical Communications, Faculty of Engineering,
Tohoku University, Sendai, Japan 980,

and

JAMES OXLEY

Department of Mathematics, Louisiana State University,
Baton Rouge, Louisiana 70803, U.S.A.